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RATIONALITY OF SECONDARY CLASSES

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Abstract

We prove the Bloch conjecture : $c_2(E) \in H^4_{\mathcal{D}}(X, \mathbb{Z}(2))$ is torsion for holomorphic rank-two vector bundles E with an integrable connection over a complex projective variety X. We prove also the rationality of the Chern-Simons invariant of compact arithmetic hyperbolic three-manifolds. We give a sharp higher-dimensional Milnor inequality for the volume regulator of all representations to $PSO(1,n)$ of fundamental groups of compact n-dimensional hyperbolic manifolds, announced in our earlier paper.

1. The theorem

1.1. Let X be a smooth complex projective variety. Consider a representation $\rho : \pi_1(X) \to SL(2, \mathbb{C})$. Let E_ρ be the corresponding rank-two vector bundle over X. Viewing E_ρ as an algebraic vector bundle, denote by $c_2(E_\rho)$ the second Chern class in Deligne cohomology group $H^4_{\mathcal{D}}(X, \mathbb{Z}(2))$ ([15], [20]). Recall that there is an exact sequence $0 \to J^2(X) \to H^4_{\mathcal{D}}(X, \mathbb{Z}(2)) \to H^4(X, \mathbb{Z}(2))$, and by the Chern–Weil theory, the image of $c_2(E_\rho)$ in $H^4(X, \mathbb{Z}(2))$ is torsion. Therefore $c_2(E_\rho)$ lies in the image of $H^3(X, \mathbb{C}/\mathbb{Z})$ under the natural map $H^3(X, \mathbb{C}/\mathbb{Z}) \to H^3(X, \mathbb{C}/\mathbb{Z}(2)) \to H^4_{\mathcal{D}}(X, \mathbb{Z}(2))$. It was proved by Bloch [3], Gillet–Soulé [24] and Soulé [50] that in fact, $c_2(E_\rho)$ is an image of the secondary characteristic class $Ch(\rho)$ of a flat bundle E_ρ (equivalently, of a representation ρ), lying in $H^3(X, \mathbb{C}/\mathbb{Z})$.The $\mathbb{R}/\mathbb{Z}$-part of this class was introduced and studied by Chern–Simons [9] and Cheeger–Simons [8],and will be called Cheeger–Chern–Simons class and denoted $ChS(\rho)$.The $\mathbb{R}$-part lying in $H^3(X, \mathbb{R})$ will be called Borel hyperbolic volume class (regulator) and denoted $Vol(\rho)$. Remark that if ρ is unitary, then $Vol(\rho) = 0$. Next, for a field F denote $\mathcal{B}(F)$ the Bloch group of F. Recall that for F algebraically closed there is an exact sequence $0 \to \mu_F^{\otimes 2} \to H_3(SL(2, F), \mathbb{Z}) \to \mathcal{B}(F) \to 0$ of Bloch–Wigner–Dupont–Sah [19]. The dilogarithm function of Bloch–Wigner defines a homomorphism $D : \mathcal{B}(\mathbb{C}) \to \mathbb{C}/\mathbb{Q} = \mathbb{R}/\mathbb{Q} \oplus i\mathbb{R}$ which splits to

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the Borel hyperbolic volume regulator and the Bloch–Beilinson Chern-Simons regulator maps [18]. Bloch [3] proved that for ρ unitary, the reduction of $Ch(\rho)$ mod $\mathbb{Q}$ lives in $\frac{Span(ReD(\mathcal{B}(\bar{\mathbb{Q}}))}{\mathbb{Q}}$, hence $c_2(E_\rho)$ assumes at most countably many values in $H^4_{\mathcal{D}}(X, \mathbb{Z}(2))$.

In [3], Bloch went on to conjecture the following result, which will be proved here.

Main Theorem A. (The Bloch conjecture). *For any representation $\rho : \pi_1(X) \to SL(2, \mathbb{C})$, the class $c_2(E_\rho) \in H^4_{\mathcal{D}}(X, \mathbb{Z}(2))$ is torsion.*

The proof of the Main Theorem A will be completed in the section 4. The strategy which we choose is described in the following lines.

1.2. The main ingredients of the proof are: the rational algebraic K-theory, the homological finiteness of S-arithmetic groups, the existence theory of twisted harmonic maps, and the Siu–Sampson–Carlson–Toledo rigidity theory.

First, we use the above-cited work of Bloch [3], Gillet–Soulé [24] and Soulé [50] to reduce the theorem to the following two statements: for any ρ as above i) $Vol(\rho) = 0$ and ii) $ChS(\rho) \in H^3(X, \mathbb{Q}/\mathbb{Z})$. Next, we consider the representation variety $Hom(\pi_1(X),\ SL_2) = V_X$. This is a scheme over $Spec(\mathbb{Z})$. Let $V_X(\rho)$ be the irreducible component of $V_X(\mathbb{C})$, containing ρ. Then $V_X(\rho)$ contains a $\bar{\mathbb{Q}}$-point, say $\bar{\rho}$. Since $V_X(\rho)$ is connected in the classical topology, the rigidity of the Chern–Simons and Borel classes [9] gives $Vol(\bar{\rho}) = Vol(\rho)$, $ChS(\bar{\rho}) = ChS(\rho)$. So we may assume that ρ is defined over an S-arithmetic groups $\mathcal{O}_S \subset F$, where F is a number field. Consider the induced map $\hat{\rho} : X \to BSL_2(\mathcal{O}_S)$. Using the work of Borel and Serre [6] on the finiteness properties of S-arithmetic groups, we will show that there exists a universal class $C : H_3(SL_n(\mathcal{O}_S)) \to \mathbb{R}$, $n \geq 2$, and a natural number M such that $M \cdot ChS(\rho) = \hat{\rho}^*(C)$, (mod $\mathbb{Z}$)).

Let $\sigma_1, \dots \sigma_m$ be the maximal set of nonconjugate embeddings of F into $\mathbb{C}$. Let $Vol \in H^3(BSL(\mathbb{C}),\ \mathbb{R})$ be the universal Borel hyperbolic volume regulator. Then by [4], the group $H^3(BSL_n(\mathcal{O}_S), \mathbb{R})$ is freely generated by $\sigma_i^* Vol$, $i = 1, \dots m$,for $n \geq 12$. Using the theory of harmonic sections and a version of the degeneration result of Sampson, we will prove that $Vol(\sigma_i \circ \rho)$ vanishes for all i. Hence $\hat{\rho}_*(H_3(X, \mathbb{Z})) \subset H_3(SL_n(\mathcal{O}_S))$ is in the torsion part of $H_3(SL_n(\mathcal{O}_S))$, so $M \cdot ChS(\rho)$ is zero in $\mathbb{C}/\mathbb{Z}$, and $ChS(\rho) \in H^3(X, \mathbb{Q}/\mathbb{Z})$.

Using a differential geometrical argument (the theory of Gromov' simplicial volume invariant) we will prove in the Section 5 the following sharp higher Milnor inequality for volume invariants, announced in [41].

Theorem B. *Let M be a compact n-dimensional hyperbolic manifold* *and let* $\mu : \pi_1(M) \to PSO(1,n)$ *be a representation. Then*

$$Vol(\mu) \leq Vol(M).$$

We will also deduce the following result whose evidence was based on the computations of Fintushel and Stern [21], and Kirk and Klassen [32]:

Theorem C. *Let* M^3 *be a Seifert fibration, and let* $\rho : \pi_1(M) \to SL_2(\mathbb{C})$ *be a representation. Then* $(ChS(\rho), [M]) \in \mathbb{R}/\mathbb{Z}$ *is rational.*

Finally, we will prove the following result.

Theorem D. *Let* M^3 *be a compact arithmetic hyperbolic manifold and let* $\rho : \pi_1(M) \to PSL_2(\mathbb{C})$ *be the defining representation. Then* $(ChS(\rho), [M])$ *is rational.*

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2. Reduction to S-arithmetic groups

2.1. Let $X/\mathbb{C}$ be as in 1.1. and let $\rho : \pi_1(X) \to SL_n(\mathbb{C})$ be a representation. The following result was proved by Bloch [3], Gillet-Soulé [24], and Soulé [50]:

Theorem. *The Chern class* $c_i(E_\rho) \in H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$ *is the image of the secondary class* $Ch(\rho) \in H^{2i-1}(X, \mathbb{C}/\mathbb{Z})$ *under the natural map* $H^{2i-1}(X, \mathbb{C}/\mathbb{Z}) \to H^{2i-1}(X, \mathbb{C}/\mathbb{Z}(i)) \to H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$.

2.2. For any finitely-generated group Γ and any algebraic group $G/\mathbb{Q}$ let V^G_Γ be the representation variety $Hom(\Gamma, G)$. This is an affine scheme defined over $Spec(\mathbb{Q})$, [12], [29]. Hence any irreducible component of $V_\Gamma(\mathbb{C})$ contains a $\bar{\mathbb{Q}}$-point.For $G = GL_n$ and $\rho : \Gamma \to GL_n(\mathbb{C})$ let $V_\Gamma(\rho)$ be the component, containing ρ and let $\bar{\rho}$ be a $\bar{\mathbb{Q}}$-point in $V_\Gamma(\rho)$.By the rigidity of the secondary classes [9] we have $ChS(\rho) = ChS(\bar{\rho})$ and $Vol(\rho) = Vol(\bar{\rho})$. So for proving the Main Theorem we may assume that ρ is defined over $\bar{\mathbb{Q}}$ and we need to show that for any singular manifold $i : M^3 \to X$, $(Vol(\rho \circ i_*), [M]) = 0$ and $(ChS(\rho \circ i_*), [M])$ is rational. Since $\pi_1(X)$ is finitely-generated, $\bar{\rho}$ is actually defined over $\mathcal{O}_S$, where S is a finite set of primes.

2.3. Let $BSL_n(\mathcal{O}_S)$ be the classifying space of $SL_n(\mathcal{O}_S)$. A representation $\rho : \pi_1(X) \to SL_n(\mathcal{O}_S)$ determines a homotopy class of maps $\hat{\rho} : X \to BSL_n(\mathcal{O}_S)$, and conversely. By the deep theorem of Borel and Serre [6], $H_*(BSL_n(\mathcal{O}_S)$ is finitely generated. In the following section we will develop for flat bundles over such spaces a general theory of regulators, and show the existence of the universal ChS-class in $H^3(BSL_n(\mathcal{O}_S), \mathbb{R}/\mathbb{Z})$.

Observe that one can define a universal class in $H^3(BSL_n^\delta(\mathbb{C}), \mathbb{R}/\mathbb{Z})$ following Beilinson and Soulé, by using the natural map of simplicial schemes $BSL_n^\delta(\mathbb{C})_. \to BSL_n(\mathbb{C})_.$. Indeed, $H^{2n}(|BSL_n(\mathbb{C})_.|)$ has pure (n,n)-type [15], and so the usual Chern class of the classifying bundle lifts to Deligne cohomology. However, we need a by-hand construction of 3.3. below, since it fits the geometric framework, in which various regulators are constructed in 3.2.1.–3.2.4. This paves the way for our use of heavy analytical weapons in Section 4, and ultimately is one of the most important components of the success in proving the Bloch conjecture. The interpretation of the Chern-Simons regulator and the Borel hyperbolic volume regulator given (within the scope of a more general theory) below in Section 3, was largely inspired by the theory of characteristic classes of foliation, developped by Bott-Haefliger and Bernstein-Rosenfeld in parallel with the ground-breaking work of Chern and Simons.

3. General theory of regulators

3.1. Let $\mathcal{CW}$ denote the category of CW-complexes and let $\mathcal{H} : \mathcal{CW} \to \mathcal{A}b$ be a cohomology theory. A functor $\mathcal{Z} : \mathcal{CW} \to \mathcal{S}ets$ and a morphism $[\ ]$: $\mathcal{Z} \to \mathcal{H}$ form a cocycle theory for $\mathcal{H}$. If for any $X \in \mathcal{CW}$ the presheaf $U \mapsto \mathcal{Z}(U)$ on X is a sheaf, we say that the theory $(\mathcal{Z}, [\ ])$ is infinitesimal. For example, the theory $X \mapsto$ (maps from X to $K(\pi, 1)$) is infinitisemal, and $X \mapsto$ (singular cocycles on X) is not, both attached to the singular cohomology theory. For a subcategory C^∞ of smooth manifolds we have a cocycle theory $\Omega : X \mapsto$ (closed exterior forms on X), which is infinitesimal.

Next, let $\pi_1(Y) \xrightarrow{\rho} Homeo(X)$ be a representation, and let $z \in \mathcal{Z}(X)$ be a cocycle, invariant under $\rho(\pi_1(Y))$. Consider the flat bundle $X \to F_\rho \to Y$, where $F_\rho = \tilde{Y} \underset{\pi_1(Y)}{\times} X$. If Y is locally simply-connected, this bundle is locally trivial. Let $\bigcup_i U_i = X$ be a covering of X by opens, such that $\pi_1(U_i) = 0$, so that $F_\rho|U_i$ is trivial. Fix an identification

$F_\rho|U_i \overset{\alpha_i}{\to} U_i \times X$ coming from the flat connection. Denote p_2: $U_i \times X \to X$ the projection to the second factor. This induces an element $(\alpha_i \circ p_2)^* z \in \mathcal{Z}(F_\rho|U_i)$, denoted y_i. Since z is invariant under $\rho(\pi_1(Y))$, these y_i form a compatible family, so if $\mathcal{Z}$ infinitesimal, there is a well-defined element $y \in \mathcal{Z}(F_\rho)$.

Assume that X is contractible. Then all sections $s : Y \to F_\rho$ are homotopic, and we obtain a well-defined element $[s^*y] \in \mathcal{H}(X)$ called the regulator of ρ and denoted $r(z,\rho)$. If X is not contractible, then anyway we obtain an element $[y] \in \mathcal{H}(F_\rho)$ and use the spectral sequence of the fibration $X \to E_\rho \to Y$ to get the secondary invariants of ρ.

3.2. Examples

3.2.1. Classical Borel regulators. Let G be a real Lie group and let $K \subset G$ be a maximal compact subgroup of G. For any manifold Y and a representation $\rho : \pi_1(Y) \to G$ we get a homomorphism $\wedge_K^*(\mathfrak{g}/\mathfrak{k}) \overset{Bor}{\to} H^*(Y)$, described as follows. Let $x \in \wedge_K^*(\mathfrak{g}/\mathfrak{k})$. Define a G-invariant form on G/K, corresponding to x and use the construction of 3.1. In particular, if $(G,K) = (SL_n(\mathbb{C}), SU(n))$, then the image of the properly normalized generator of $\wedge_K^3(\mathfrak{sl}_n/\mathfrak{su}(n))$ is called the hyperbolic volume of ρ, denoted $Vol_3(\rho)$. For the pair $SO(1,n)$, $SO(n))$ we get an element $Vol_n(\rho)$ corresponding to the generator of $\wedge^n(\mathfrak{so}(1,n)/\mathfrak{so}(n))$, also called the hyperbolic volume; see [41], for example.

3.2.2. Cheeger–Chern-Simons classes. Again let G be a real Lie group, and let $x \in \wedge^*(\mathfrak{g})$. Let z be a left-invariant form in $\Omega^*(G)$, corresponding to G. Then for any manifold Y and any representation we get an element $[y] \in H^*(F_\rho)$. If G is contractible, e.g. $G = \tilde{SL}_2(\mathbb{R})$, this defines an element $r(x,\rho) \in H^*(Y,\mathbb{R})$ as above. We refer to [44], to the detailed study of this last example. If G is not contractible, one looks at the spectral sequence of F_ρ to see what can be done to descend some cohomology information down to Y.

Let us specialize this construction for the Cartan form $\omega(X, Y.Z) = ([X,Y],Z)$, where $(\cdot,\cdot)$ is the Cartan-Killing scalar product in $sl_n(\mathbb{C})$. This is a complex-valued invariant 3-form, so that we may look at $r(Re\ \omega,\rho)$ and $r(Jm\ \omega,\rho)$ both in $H^3(F_\rho,\mathbb{R})$. Observe that F_ρ here is a flat principal $SL_n(\mathbb{C})$-bundle over Y. Now, since $Jm\ \omega$ is exact, we may descent $r(Im\ \omega,\rho)$ to $H^3(Y,\mathbb{R})$.We claim that this will be precisely the hyperbolic volume regulator of 3.2.1. Indeed, fix a point $p \in \mathcal{H}^3$and consider the evaluation map from $SL(2,\mathbb{C})$ to $\mathcal{H}^3$, sending g to gp.This map is equivariant with respect to $SL(2,\mathbb{C})$- actions considered, the pull-back of the volume form on $\mathcal{H}^3$ is precisely $Jm\ \omega$, and we can used the functoriality by X in 3.1. On the other hand, $Re\ \omega$ is not exact and represents a generator of $H^3(SL_n(\mathbb{C}),\mathbb{R}) \approx \mathbb{R}$. Normalizing it to

$\frac{1}{4\pi^2} Re\ \omega$ so that its period will be one, we easily see, in the case where F_ρ is topologically trivial, that the descent by different sections will give a well-defined element in $H^3(Y, \mathbb{R}/\mathbb{Z})$. This is the classical Chern-Simons class. If F_ρ is not topologically trivial, then it is still well-defined [9].

3.2.3. Thurston-MacDuff-Morita invariants. Let X be a contractible manifold with a volume form ω; e.g., $(X, \omega) = (\mathbb{R}^n, can)$. Then for any manifold Y and a representation $\rho : \pi_1(Y) \to Diff_\omega(X)$ one applies 3.1 to get an element $Bor(\omega, \rho) \in H^n(Y, \mathbb{R})$. It is easily shown that this element comes from the universal class $Vol_\omega \in H^n\ (Diff_\omega(X), \mathbb{R})$. Similarly let (X, σ) be a contractible symplectic manifold. Then there exists an element $Sympl \in H^2(BSympl\ (X),\ \mathbb{R})$, where $Sympl\ (X)$ is the symplectomorphism group. These classes were defined in [34], [35], [54], [31], using simplicial constructions. A somewhat deeper look at the topology of symplectic fibrations enables one to define a Chern-Simons-type invariants; c.f. [46]. This uses the invariant polynomials on the Lie algebra of $Sympl(X)$, introduced and studied in [46].One may say that "cohomologically" $Sympl(X)$ behaves as a finite-dimensional Lie group.We refere to [46] for the details.

3.2.4. K-theoretic invariants of group actions. Let Γ be a group acting smoothly on a manifold X. Form a flat bundle $X \to F \to B\Gamma$ and consider a vector bundle $\mathcal{F}$ over F, tangent to the fibers. This is well-defined despite the fact that $B\Gamma$ is not a manifold. If X is contractible, find a section s of F and consider the class $s^*[\mathcal{F}] \in K^0(B\Gamma)$, which is a well-defined invariant of the action. If X is not contractible we may look at the characteristic class of $\mathcal{F}$ in the singular cohomology of F and try to get the secondary invariants in $H^*(\Gamma)$, using the spectral sequence, as above. This construction does not fall under the axiomatic description of 3.1. since the cocycle theory $X \mapsto$ (isomorphic classes of vector bundles over X) is not infinitesimal. The problem of finding better axiomatic description, covering this case, is left for the interested reader.

The secondary classes of group actions, described above, prove important in applications to the finite group actions. Details will appear elsewhere.

3.3. In this paragraph we will describe the construction of regulators for the de Rham cocycle theory in the case where Y is a locally simply connected CW–complex, not a manifold (compare [18]). We concentrate on the special case $X = SL_n(\mathbb{C})$, always assume that $H_*(Y)$ is of finite type. Then the image of the natural map $MSO_k(Y) \to H_k(Y)$ is of finite index in $H_k(Y)$ by the theorem of Thom. For any representation $\rho : \pi_1(Y) \to SL_n(\mathbb{C})$ the usual Chern-Weil theory will then imply that all Chern classes $c_i(E_\rho) \in H^{2i}(Y,\mathbb{Z})$ die after tensoring by $\mathbb{R}$, hence $c_i(E_\rho) \in H^{2i}_{tors}(Y,\mathbb{Z})$. Let N be such that $Nc_2(E_\rho) = 0$. Then $c_2(E_\rho)$ is a Bockstein image of a class z in $H^3(Y,\mathbb{Z}/N\mathbb{Z})$. Let $f : Y \to K(3,\mathbb{Z}/N\mathbb{Z})$ be the classifying map and let $\bar{Y}$ be a homotopical fiber of f. Then the inclusion $i : \bar{Y} \to Y$ induces an isomorphism in the rational homology. Pull back the flat bundle E_ρ from Y to $\bar{Y}$and denote $\bar{E}_\rho$ the resulting bundle. Then $c_2(\bar{E}_\rho) = 0$, hence the principal $SL_n(\mathbb{C})$-bundle $\bar{F}_\rho$ associated to $\bar{E}_\rho$ admits a section over the 4-skeleton $Sk_4(\bar{Y})$. Fix such a section s. Let $\bar{\rho} = \rho \circ i_* : \pi_1(\bar{Y}) \to SL_n(\mathbb{C})$. Then $\bar{E}_\rho$ is a flat bundle, associated with $\bar{\rho}$. Consider a singular manifold $j : M^3 \to Sk_4\bar{Y}$ and a flat bundle $j^*\bar{E}_\rho$ with the canonical *smooth* structure (of a flat bundle). Consider the 3-form $x = Re([X,Y],Z)$ on $sl_n(\mathbb{C})$, where $(\cdot,\cdot)$ is the (complex) Cartan-Killing scalar product. The corresponding regulator $r(x,j^*\rho) \in H^3(j^*\bar{F}_\rho,\mathbb{R})$ is the usual Chern-Simons invariant. Next, we use *the canonical* choice of a section, namely, $s \circ j$, to produce a class, also called $r(x,j^*\rho)$, in $H^3(M,\mathbb{R})$ and a number $(r(x,j^*\rho),\ [M]) \in \mathbb{R}$. We claim this defines a homomorphism $MSO_3(\bar{Y}) \to \mathbb{R}$. Indeed, suppose we are given a map $\psi : N^4 \to \bar{Y}$ with $(M,j) = \partial(N,\psi)$. Arguing as above, we get a class $r(x,\psi^*\rho) \in H^3(\psi^*\bar{F}_\rho,\mathbb{R})$, whose restriction on $\phi^*\bar{F}_\rho|_{\partial N}$ gives $r(x,j^*\rho)$. Then it is obvious that the latter class is zero.

Since $H_*(\bar{Y})$ is finitely generated, there is a homomorphism $H^3(\bar{Y}) \approx H_3(Sk_4\bar{Y}) \to \mathbb{R}$, whose reduction mod $\mathbb{Z}$ induces a usual Chern-Simons class on every singular manifold $j : M^3 \to \bar{Y}$. Now, since $i_* : H_3(\bar{Y},\mathbb{Q}) \xrightarrow{\sim} H_3(Y,\mathbb{Q})$, we have a homomorphism $C : H_3(Y,\mathbb{Z}) \to \mathbb{R}$, such that for some number $M \in \mathbb{N}$ large enough the following is true: for any singular manifold $j : M^3 \to Y$, $\quad C(j_*[Y]) = ChS(j \circ \rho) \pmod{\mathbb{Z} \cdot \frac{1}{M}}$.

3.4. Remark. As it was mentioned above,there exists a universal ChS class in $H^3(BSL_2(\mathbb{C}),\mathbb{R}/\mathbb{Z}))$.

4. Proof of the Main Theorem

4.1. Let F be a number field without real places, and let $\sigma_1,\dots\sigma_m$ be a maximal family of nonconjugate embeddings of F into $\mathbb{C}$. Let $\mathcal{O}_S$ be as above. Consider the universal Borel regulator $Vol \in H^3(BSL_n(\mathbb{C}),\mathbb{R})$. We need the following fundamental result of Borel [4].

Theorem. (A. Borel) *The elements* $\sigma_i^* Vol \ \in H^3(BSL_n(\mathcal{O}_S),\mathbb{R})$ *form a basis of* $H^3(BSL_n(\mathcal{O}_S),\mathbb{R})$ *over* $\mathbb{R}$ *for* n *large enough* ($n \geq 12$).

4.2. Combining Theorem 4.1 with 3.3 gives the following:

Fundamental Lemma. *There exist constants* $\alpha_1,\dots\alpha_m \in \mathbb{R}$ *and a natural number* M', *such that for any representation* $\rho : \pi_1(M^3) \to$

$SL_n(\mathcal{O}_S)$,

$$(*) \qquad ChS(\rho) \equiv \sum_{i=1}^{m} \alpha_i Vol(\sigma_i \circ \rho) \pmod{\mathbb{Z} \cdot \frac{1}{M'}},$$

where n is as in 4.1.

Since the ChS-invariant is compatible with the embeddings $SL_m \to SL_n$, $n > m$, we can remove the restriction on n.

4.3. To prove Theorem A we need to show that $Vol(\mu) = 0$ for any representation $\mu : \pi_1(X) \to SL_2(\mathbb{C})$. Applying this to $\mu = \sigma_i \circ \rho$ we will get $ChS(\rho) \in \mathbb{Z} \cdot \frac{1}{M'}$ on $MSO_3(X)$ by 4.2.

Consider the natural action of $SL_2(\mathbb{C})$ on the hyperbolic space $\mathcal{H}^3$. Let S_∞ be the sphere at infinity. We may assume that $\mu(\pi_1(X))$ does not have fixed points in S_∞, otherwise the representation μ factors through $\mathbb{R}_+ \times (SO(2) \times \mathbb{R}^2)$ and so deformes to a representation $\bar{\mu}$ in $SO(2) \times \mathbb{R}^2$ and $Vol(\mu) = Vol(\bar{\mu}) = 0$ by 4.5.1 and the rigidity of Vol. Consider the flat bundle $\mathcal{H}^3 \to \mathcal{F} \to X$ associated to μ. We need the following fundamental result of Donaldson and Corlette [17], [13]; see also [30] and [33]. A similar result for representations in $SL_2(\mathbb{R})$ had been established earlier by Diederich and Ohsawa [16] by essentially the same argument.

Theorem. (S. Donaldson, K. Corlette) *Let X be a compact Riemannian manifold and let $\mu : \pi_1(X) \to SL_2(\mathbb{C})$ be a representation. If the action of $\pi_1(X)$ on S_∞ is fixed-point-free, then the flat bundle $\mathcal{F}$ posesses a harmonic section.*

4.4. Harmonic maps of Kähler manifolds to manifolds of negative curvature were intensively studied by many authors, starting with the seminal work of Siu [49]. We quote in particular the following remarkable result of Sampson [48].

Theorem (Sampson). *Let X be a compact Kähler manifold and Z a (real) hyperbolic manifold. Let $\phi : X \to Z$ be harmonic. Then rank $D\phi \leq 2$ everywhere on X.*

The proof of this theorem remains valid for harmonic sections of flat bundles, associated with a representation $\mu : \pi_1(X) \to Iso(Z)$, as in [41]. Indeed, the proof in [48] is based on the Bochner–type integral formula, which is local in ϕ and hence holds for harmonic sections.

4.5. Now we are ready to complete the proof of Theorem A. Let $\mu : \pi_1(X) \to SL_2(\mathbb{C})$ be a representation. Assuming that $\pi_1(X)$ acts fixed-point-free on $S_\infty(\mathcal{H}^3)$, consider a harmonic section s of $\mathcal{F}$ which exists by 4.3. Let $\omega \in \Omega^3(\mathcal{F})$ be the three-form defined by the pull-back from the volume form of hyperbolic fibers; it is well-defined by 3.1. The form $s^*\omega$ represents $r(\mu) = Vol(\mu) \in H^3(X, \mathbb{R})$. On the other hand, Ds_x, viewed as a map from T_xX to $T_{s(x)}\mathcal{F}_x$, has a rank ≤ 2 by 4.4., for all $x \in X$, so $s^*\omega$ vanishes identically. This proves $Vol(\mu) = 0$ and hence $ChS(\rho) \in H^3(X, \mathbb{Z} \cdot \frac{1}{M'}/\mathbb{Z})$ by 4.2.

4.5.1. So what is left is to show that if the representation $\mu : \pi_1(X) \to SL_2(\mathbb{C})$ factors through $SO(2) \times \mathbb{R}^2$, then $Vol(\mu) = 0$. Recall [40] that the volume invariant may be interpreted as follows. Consider the continuous cohomology $H_c^*(SL_2(\mathbb{C}))$. There exists a natural map $H_c^3(SL_2(\mathbb{C}), \mathbb{C}) \to H^3(SL_2^\delta(\mathbb{C}), \mathbb{C})$. The left-hand side space has dimension 1, and the image of the canonical generator is precisely the element $Vol \in H^3(SL_2^\delta(\mathbb{C}))$. Now, we obtain a diagram:

$$\begin{array}{ccc} H_c^3(SL_2(\mathbb{C}), \mathbb{C}) & \longrightarrow & H_c^3(SO(2) \times \mathbb{R}^2)^\delta, \mathbb{C}) \\ \downarrow & & \downarrow \\ H^3(SL_2^\delta(\mathbb{C}), \mathbb{C}) & \longrightarrow & H^3((SO(2) \times \mathbb{R}^2)^\delta, \mathbb{C}) \end{array}$$

Next, since $SO(2)$ is compact and $\mathbb{R}^2$ is acyclic we have the canonical isomorphism [21] $H_c^3(SO(2) \times \mathbb{R}^2, \mathbb{C}) \simeq H^3(\mathfrak{so}(2) \times \mathbb{R}^2, SO(2)) = 0$.

Following the suggestion of the referee, we describe a different argument of more geomerric nature.If the representation μ fixes a point at infinity, it also leaves invariant the corresponding horocycle foliation.That means that the flat fibration $\mathcal{F}$ is foliated by the flat subfibrations with horocycles as fibers.Since horocycles are contractible, we can find a section s which stays in any of the horocycle subfibrations. But then again Ds_x has rank at most two, so $Vol(\mu)$ is zero.

4.6. Assume that X is a compact Kähler manifold and $\rho : \pi_1(X) \to SL_2(\mathbb{C})$ is a representation. Then $ChS(\rho)$ is rational. The proof is identical to that of Theorem A. Alternatively, for discrete representations one can use the Theorem A and the result of Mok [38] which states that any such representation ρ factors through a homomorphism $\pi_1(X) \to \pi_1(Z)$ with Z smooth and projective. On the other hand, Theorem A probably fails for compact complex manifolds X in view of the recent result of Taubs, saying that any finitely presented group is a complex manifold group.

The result on vanishing the $Vol(\mu)$, where $\mu : \pi_1(X) \to SL_2(\mathbb{C})$, is not true for representations in Lie groups of higher rank. However, there are important results on the rigidity of representations with prescribed higher dimensional volume [14]. This also uses the harmonic section technique. Another interesting application is a construction of a huge family of compact symplectic manifolds, which do not admit any Kähler structure and have a fundamental group of an exponential growth. See [43] for the details.

4.7. The main theorem above provokes a natural question: what happens for flat bundles of higher rank? Because of what we have just observed, it is clear that the proof presented here does not apply directly to this situation. Still, one can expect the validity of the following statement.

Bloch conjecture (higher rank case). Let X be a complex projective variety and let $\rho : \pi_1(X) \to SL_n(\mathbb{C})$, $n \geq 2$, be a representation. Let E_ρ be the corresponding flat rank n vector bundle.Then for all $i \geq 2$, $c_i(E_\rho) \in H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$ is torsion.

As stated, this conjecture becomes very similar to the generalized Bloch-Beilinson conjecture on higher regulators of motives. Anyway, this conjecture is true; the detailed proof and all the yoga around it will be presented in the forthcoming paper [45].

On the other hand, for $i = 1$ and the representations in $GL_n(\mathbb{C})$ one cannot expect anything like this to hold, just because even for curves, the first Chern-Simons class of a flat line bundle is just its monodromy presentation $\pi_1(X) \to \mathbb{C}^*$, which may be completely arbitrary. However, if X is defined over a number field k and if the flat bundle comes from the Deligne-Ramakrishnan construction, associated to an element z of $K_2(X_k)$ [40], one expects a lot of rigidity for the monodromy representation. The celebrated Bloch-Beilinson conjecture relates its periods to the value of L-function of X at $s = 0$. Moreover, the logarithms of the real parts of these periods are believed to form a $\mathbb{Q}$-structure of $H^1(X, \mathbb{R})$ if $[k : \mathbb{Q}] = 1$ and z varies inside $K_2(X_{\mathbb{Q}}) \otimes \mathbb{Q}$. The reader will find the results of our geometrical treatment of higher regulators in algebraic K–theory in a forhcoming paper.

5. Higher Milnor inequality and ChS invariants of Seifert fibrations

5.1. In this section we will prove Theorems B and C. The proof will be based on the sharp higher Milnor inequality, announced in [41] with the proof of a weaker estimate given there. For a manifold M we denote $||M||_g$ the Gromov's simplicial volume of the fundamental cycle. Let $\mu : \pi_1(M) \to PSO(1, n)$ be a representation. Since $PSO(1, n)$ acts isometrically in $\mathcal{H}^n$, the general theory of 3.1 gives us an invariant $Vol(\mu) \in H^n(M, \mathbb{R})$. For n-dimensional M we denote $(Vol(\mu), [M])$ again by $Vol(\mu)$. Then we state the following result, whose proof will be completed in 5.19.

Theorem. *For any compact n-dimensional manifold M and any representation $\mu : \pi_1(M) \to PSO(1, n)$, the volume $Vol(\mu)$ satisfies*

$$Vol(\mu) \leq d_n ||M||_g,$$

where d_n is the Milnor constant, i.e., the volume of the regular infinite simplex.

5.2. Combining 5.1. with a result of Gromov [27], [53] we get

Theorem B. (Higher Milnor Inequality). *Let $M^n, n \geq 2$ be a closed hyperbolic manifold and let $\mu : \pi_1(X) \to PSO(1,n)$ be a representation. Then*

$$Vol(\mu) \leq Vol\ M.$$

The discussion of the classical case $n = 2$ with various generalizations is to be found in [41]. The proof of 5.1, 5.2 follows the pattern given in [41].

5.3. Remark. Theorems 5.1, 5.2 are directly inspired by, and are generalizations of, the inequality of Gromov-Thurston [53] stating that for any map $\phi : M \to N$ with N hyperbolic one has

$$deg\ \varphi \cdot Vol\ N \leq d_n \cdot \|M\|_g.$$

5.4. Using 5.1. we now prove Theorem C. Let M^3 be a Seifert fibration by a theorem of Yano [57], $\|M\|_g = 0$. Hence by 5.1. $Vol(\mu) = 0$ for any μ. Then applying 4.2. we get $ChS(\rho) \in H^3(M, \mathbb{Z} \cdot \frac{1}{M'})$.

5.5. Example. Let $\Gamma \subset \tilde{SL}_2(\mathbb{R})$ be a uniform lattice. Then for some canonical choice of the Haar measure in $\tilde{SL}_2(\mathbb{R})$,

$$Vol(\tilde{SL}_2(\mathbb{R})/\Gamma) \in \mathbb{Q}.$$

Proof. We normalize the ChS form on $SL_2(\mathbb{C})$ in such a way that the period of this form (the integral over $SU_2 \subset SL_2(\mathbb{C})$) is one. For a uniform lattice $\Gamma \subset \tilde{SL}_2(\mathbb{R})$ denote $M = \tilde{SL}_2(\mathbb{R})/\Gamma$ and let $\rho : \pi_1(M) = \Gamma \to \tilde{SL}_2(\mathbb{R}) \to SL_2(\mathbb{R}) \to SL_2(\mathbb{C})$ be the canonical representation. Then Covol $(\Gamma) = ChS(\rho)$. Moreover, M is a Seifert fibration [37] and 5.4 applies to prove 5.5.

5.6. Example. Let $M = \{z_1^p + z_2^q + z_3^r = 0\} \cap S^5 \subset \mathbb{C}^3$ be a Pham-Brieskorn homology sphere, where p, q, r are coprime and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Then by Milnor and Dolgachev (see [37]), $M = \tilde{SL}_2(\mathbb{R})/\Gamma$, so that $Vol(M)$ is rational. The ChS-invariant of representations in $SL_2(\mathbb{R})$ has hyperbolicity properties studied in [44]. In particular, it is shown there that the topological complexity of $\underset{m}{\#} M$ grows at least logarithmically with m if M is a Pham-Brieskorn variety.

5.7. Example. Let $M = \sum(a_1, \dots a_n)$ be a genus-zero Sifert fibration, so a homology sphere. The representation variety $Hom(\pi_1(M), SU_2)/SU_2$ is studied by Fintushel-Stern[21], Kirk-Klassen [32], and Bauer-Okonek [1]. Fintushel and Stern [21] computed ChS invariants of all unitary representations of $\pi_1(M)$. These are rational numbers with denominator a divisor of $a = a_1 \dots a_n$. It would be interesting to compute ChS invariants of all representations in $SL_2(\mathbb{C})$.

5.8. Let M be a compact oriented three-manifold. We will say that M is **of hyperbolic type**, if there exists a representation $\mu : \pi_1(M) \to SL_2(\mathbb{C})$ with $Vol(\mu) \neq 0$. We actually have proved the following:

Proposition. *If there exists a representation $\rho : \pi_1(M) \to SL_2(\mathbb{C})$ with irrational ChS-invariant, then M is of hyperbolic type.*

Moreover, as we just have seen, all Seifert fibrations are not of hyperbolic type. We notice that the manifolds of hyperbolic type enjoy many properties of actually hyperbolic manifolds, as follows.

5.9. Proposition. (compare [27], [53]). *Let N, M be compact oriented three-manifolds such that M is of hyperbolic type. Then the degrees of continuous maps $\phi : N \to M$ are bounded by a constant $C(N, M)$.*

Proof. Let $\mu : \pi_1(M) \to SL_2(\mathbb{C})$ be a representation with $Vol(\mu) \neq 0$. For a map $f : N \to M$ by the naturality of the volume invariant we have: $Vol(\mu \circ f_*) = \deg f \cdot Vol(\mu)$. On the other hand, $Vol(\mu \circ f_*) \leq \|[N]\|_g$ by 5.1. Hence $\deg f \leq |Vol(\mu)|^{-1} \|[N]\|_g$.

5.10. Corollary. (c.f. [27],[53]). *Let M be of hyperbolic type. Then for a self-map $f : M \to M$, $\deg f$ assumes one of the values: $0, \pm 1$.*

5.11. Remarks. (i) If M is of hyperbolic type, and N is any three-manifold, then $M \# N$ is of hyperbolic type

(ii) If M is of hyperbolic type, and $\varphi : N \to M$ is a ramified covering along a link in M, then N is of hyperbolic type.

Observe that atoroidal irreducible manifolds of hyperbolic type a virtually hyperbolic by positive solution to Thurston conjectuer [47].

5.12. A well-known theorem of Goldman [25] asserts that for $n = 2$, the equality in 5.2. takes place iff the monodromy group of μ is a uniform lattice in $PSL_2(\mathbb{R})$. This generalizes to $n \geq 3$ as follows: any representation for which $Vol(\mu) = Vol(M)$ is conjugate to a composition $\rho \circ f_*$, where $\rho : \pi_1(M) \to PSO(1, n)$ is the defining representation of the hyperbolic manifold M, and $f : M \to M$ is an isometry. The proof will appear elsewhere.

5.13. Recall the following classical result of A. Weil [56] and Garland-Raghunathan [23].

Theorem. *Let M be a complete hyperbolic manifold of finite volume, $\dim M \geq 3$, and let $\rho : \pi_1(M) \to PSO(1, \omega)$ be the defining representation. Then ρ is rigid in the following cases:*

(a) M is compact.

(b) M is noncompact and $\dim M \geq 4$.

Combining this theorem with the argument of 2.2 (called the Vinberg lemma by Margulis) one get the following important

Corollary. *If M is a hyperbolic manifold with* $\dim M \geq 3$ *and either M is compact or* $\dim M \geq 4$, *then the defining representation ρ is defined over a number field. Alternatively, the lengths of all closed geodesics of M are algebraic numbers.*

This inspires the following definition.

5.14. Definition. For M as above, let $F \subset \mathbb{C}$ be the field, generated by the lengths of closed geodesics of M. Then F is called the field of definition of M. The natural number $g(M) = [F : \mathbb{Q}]$ is called the arithmetic complexity of M.

5.15. Let $\mathcal{HYP}(n), n \geq 3$, be the set of isometry classes of compact hyperbolic manifolds of dimension n. For $n \geq 4$ the volume function $Vol : \mathcal{HYP}(n) \to \mathbb{R}_+$ is proper, i.e., $\#Vol^{-1}([0, C]) < \infty$ for any threshold $C > 0$ by the famous theorem of Wang [55] and Gromov [26]; see also [42]. This fails for $n = 3$ [53]. However, we state the following conjecture.

Conjecture. *The function $g + Vol : \mathcal{HYP}(3) \to \mathbb{R}_+$ is proper. Morover, for any number field $F \subset \mathbb{C}$ there are but finitely many uniform lattices in $SL_2(\mathbb{C})$, which are contained in $SL_2(F)$.*

Observe that $g + Vol$ extendeds to all irreducible atoroidal manifolds with nonzero Casson invariants [47].

5.16. Here we show how to deduce 5.13(a) from 5.12. The conception of the proof borrows a lot from the Gromov's approach to the Mostow rigidity theorem. We start with the following lemma.

5.16.1. Lemma (Rigidity of Vol). *Let M be a compact manifold and let ρ_t, $0 \leq t \leq 1$, be a continuous family of representations of $\pi_1(M)$ in $PSO(1, n)$. Then $Vol(\rho_t) = const$.*

Proof. Fix a point p in $\mathcal{H}^n$ and consider the evaluation map $v_p : PSO(1, n) \to \mathcal{H}^n$. Let ω be the volume form in $\mathcal{H}^n$. Since the pullback $v_p^*\omega$ is a left-invariant form on $PSO(1, n)$, we may consider the regulator $r(v_p^*\omega, \mu)$ by 3.2.2. Clearly $r(v_p^*\omega, \mu) = Vol(\mu) \in H^n(M, \mathbb{R})$.

Now, let $\alpha \in \Lambda^n \mathfrak{pso}^*(1, n)$ be the element corresponding to the left-invariant closed $v_p^*\omega$. Since ω is closed, α is a cocycle for the Lie algebra cohomology. Let $\mathcal{P}_t$ be the flat principal $PSO(1, n)$-bundle, corresponding to ρ_t. We may view $\mathcal{P}_t$ as a fixed principal bundle with varying flat connection $\omega_t \in \Omega^1(\mathcal{P}, \mathfrak{pso}(1, n))$. Then $r(v_p^*\omega, \rho_t)$ may be described as the characteristic class of a $\mathfrak{pso}(1, n)$-structure on $\mathcal{P}$, corresponding to the cocycle α, in the sense of Bott-Halfliger-Bernstein-Rosenfeld [2], [7]. Since $\mathfrak{pso}(1, n)$ is a finite-dimensional semi-simple algebra, all such classes are rigid [22]. This proves the lemma.

5.16.2. Now let M be a compact hyperbolic manifold, and let

$\rho : \pi_1(M) \to PSO(1,n)$ be the defining representation. We wish to prove that ρ is rigid. Let ρ_t be a path in $V_{\pi_1(M)}^{PSO(1,n)}$ which starts with ρ. By 5.12.1., $Vol(\rho_t) = Vol(\rho)$. Then by 5.12., ρ_t is conjugate to $\rho \circ f_{t*}$ for some isometry f_t. But $Iso(M)$ is finite by the Bochner theorem, so ρ_t is conjugate to ρ,which proves 5.13 (a).

5.17. We will indicate an application of 5.16.1. to the geometry of representation varieties. Start with a hyperbolic three-manifold M. Let L be a link in M, homologeous to zero, and let $N \xrightarrow{f} M$ be a d-sheet ramified covering along L. Assume that N is itself hyperbolic (this is almost always the case by [47]). Let μ, ν be respectively the defining representations of $\pi_1(M)$, $\pi_1(N)$ in $PSL_2(\mathbb{C})$. Then by the Gromov inequality, $Vol(N)$ is strictly larger than $d \cdot Vol(N)$. Hence, by 5.16, $V_{\pi_1(N)}^{PSL_2(\mathbb{C})}$ has at least two connected components (in classical topology), containing ν and $\mu \circ f_*$, respectively. Proceeding in this way with M replaced by N, we come to the following.

Proposition. *There exists an irreducible compact three-manifold P, such that the representation variety $V_{\pi_1(p)}^{PSL_2(\mathbb{C})}$ has an arbitrarily large number of components.*

5.18. If M is a noncompact hyperbolic three-manifold, then 5.13(b) fails to be true. Moreover, the well-known result of Thurston (see [12]) estimates the dimension of $V_{\pi_1(M)}(\rho)$ by the number of cusps in M. We may alter Definition 5.14. as follows: by 5.2. $V_{\pi_1}(\rho)$ contains a $\bar{\mathbb{Q}}$-point $\bar{\rho}$. Let F be a field of the smallest degree such that there exists a F-point in $V_{\pi_1(M)}(\rho)$; put $g(M) = [F : \mathbb{Q}]$. In particular, for an excellent knot $K \subset S^3$ this defines **the arithmetic complexity of the knot** K when applied to the (hyperbolic of finite volume) knot manifold $S^3 \backslash K$.

5.19. We begin a proof of 5.1. Consider the flat $\mathcal{H}^n$-bundle $\mathcal{F}$ over M, corresponding to μ, that is, $\mathcal{F} = \tilde{M} \underset{\pi_1(M)}{\times} \mathcal{H}^n$, where $\pi_1(M)$ acts in $\tilde{M} \times \mathcal{H}^n \equiv \mathcal{H}^n \times \mathcal{H}^n$ by the diagonal action (ρ, μ). Fix a section, say s, of $\mathcal{F}$. By the well-known relation between sections and equivariant maps (see [11], for example), this gives rise to an equivariant map $\bar{s} : \tilde{M} \to \mathcal{H}^n$ with respect to the actions ρ and μ. Let $\Sigma a_i \sigma_i$ be a closed singular chain in M. Let $\tilde{\sigma}_i$ be a lift of σ_i to $\tilde{M} = \mathcal{H}^n$. Let $\hat{\sigma}_i$ be the Thurston straightening of the singular simplex $\bar{s}(\tilde{\sigma}_i)$. Consider the chain $\Sigma a_i(\tilde{\sigma}_i, \hat{\sigma}_i)$ in $\mathcal{H}^n \times \mathcal{H}^n$. Denote by $p : \mathcal{H}^n \times \mathcal{H}^n \to \mathcal{F}$ the natural projection and consider the chain $b = \Sigma a_i p(\tilde{\sigma}_i, \hat{\sigma}_i)$ in $\mathcal{F}$. Let $\pi : \mathcal{F} \to M$ be the fibration map. Then $\pi(b) = \Sigma a_i \sigma_i$. We claim that b is closed. This follows immediately from the description of the straightening process (see [53]) and the fact that $\Sigma a_i \sigma_i$ is closed in M. Next, let ω be the volume form in $\mathcal{H}^n$ and let $\pi_2 : \mathcal{H}^n \times \mathcal{H}^n \to \mathcal{H}^n$ be the projection on the second factor. Then clearly $Vol(\mu) = \int\limits_{\Sigma_i(\bar{\sigma}_i, \hat{\sigma}_i)} \pi_2^* \omega \leq \Sigma |a_i| \cdot d_n$ where d_n is the Milnor constant. Taking infinum over all chains representing [M], we get $Vol(\mu) \leq \|[M]\|_g$.

6. The Chern-Simons invariant of arithmetic hyperbolic three-manifolds

6.1. In this section we will prove Theorem D. Recall that all arithmetic hyperbolic three-manifolds are constructed as follows. Let F be a totally real number field, and let Q be a quadratic form in four variables, defined over F. Suppose that Q has the signature (1.3) and that for any nontrivial embedding $\sigma : F \to \mathbb{R}$ the form Q^σ is negatively defined. Then $SO(Q) \cap SL_4(\mathcal{O}) \subset SO(1,3)$ is a uniform lattice.

6.2.1. Fix the identification $PSL_2(\mathbb{C}) \approx SO(1,3)$. It follows that the defining representation ρ of an arithmetic hyperbolic three-manifold M is defined over $K = F[\sqrt{-1}]$ where F is totally real. In particular, all embeddings $\sigma : K \to \mathbb{C}$ commute with the complex conjugation. Let $\{\sigma_i\}$ be the maximal family of nonconjugate embeddings. Then by 4.2., for any representation $\mu : \pi_1(M) \to SL_2(\mathbb{C})$ we have

$$(*) \qquad ChS(\mu) = \sum_{i=1}^{m} \alpha_i Vol(\sigma_i \circ \mu) \quad (\mathrm{mod} \mathbb{Z} \frac{1}{M'}).$$

Since the defining representation $\rho : \pi_1(M) \to PSL_2(\mathbb{C})$ lifts to $SL_2(\mathbb{C})$ [52]. (*) applies to ρ without any change, so

$$(**) \qquad ChS(\rho) = \sum_{i=1}^{m} \alpha_i Vol(\sigma_i \circ \rho) \quad (\mathrm{mod} \mathbb{Z} \frac{1}{M'}).$$

Now, applying (*) to the complex-conjugate representation $\bar{\rho}$ we get

$$(***) \quad ChS(\bar{\rho}) = \sum_{i=1}^{m} \alpha_i \, Vol(\sigma_i \circ \bar{\rho}) = \sum_{i=1}^{m} \alpha_i \, Vol(\overline{\sigma_i \circ \rho})(\mathrm{mod} \mathbb{Z} \cdot \frac{1}{M'}),$$

since σ_i commutes with the conjugation. Now we will use the following lemma.

6.2.2. Lemma. *For any representation $\mu : \pi_1(M) \to PSL_2(\mathbb{C})$ we have*

(i) $ChS(\bar{\mu}) = ChS(\mu) \ (\mathrm{mod} \mathbb{Z})$,
(ii) $Vol(\bar{\mu}) = -Vol(\mu)$.

Proof. Let $\omega(X, Y, Z) = ([X, Y], Z)$ be the canonical 3-form in $SL_2(\mathbb{C})$. Then by 3.3., $ChS(\mu) = r(Re\ \omega,\ \mu)$ whereas $Vol(\mu) = r(Im\ \omega, \mu)$, and the statement of the lemma follows readily.

6.2.3. To finish the proof of Theorem D, we add (**) and (***) to get $2ChS(\rho) = \sum \alpha_i(Vol(\sigma_i \circ \rho) + Vol(\overline{\sigma_i \circ \rho})) = 0 \pmod{\mathbb{Z} \cdot \frac{1}{M}}$, hence $ChS(\rho) \in \mathbb{Q}$.

6.3. Remarks. What we actually have proved is $2M' \cdot ChS(\rho) \in \mathbb{Z}$, where M is defined by **3.3**. The inspection of 3.3. shows that M' is just the squared order of the (torsion) second Chern class of the classifying bundle over $BSL_2(F[\sqrt{-1}])$. The latter invariant was intensively studied in algebraic K-theory (see [51], [52] for the references therein and the connection to the Lichtenbaum conjecture).

6.3.1. The arithmetic hyperbolic three-manifolds constitute relatively "small" part of all hyperbolic manifolds; in particular, the set of volumes of these manifolds is discrete [10], [5]. However, Corollary 5.13. shows that in a way *all* compact hyperbolic manifolds are "arithmetic". A somewhat deeper look at the proof of Theorem D above indicates at the level of difficulties in studying the general case. If the number field of definition is not a CM field, then the complex conjugation permutes all the embeddings into $\mathbb{C}$ in a way we may not control from the point of view of the arithmetic nature of the coefficients α_i in (*). These coefficients depend only on the number field and are very interesting invariants of it.

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